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Derivation of Nonlinear Schrödinger Equation

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Abstract We propose some nonlinear Schrödinger equations by adding some higher order terms to the Lagrangian density of Schrödinger field, and obtain the Gross-Pitaevskii (GP) equation and the logarithmic form equation naturally. In addition, we prove the coefficient of nonlinear term is very small, i.e., the nonlinearity of Schrödinger equation is weak.

Keywords Nonlinear Schrödinger equation · Lagrangian density · Higher order term

1 Introduction

Quantum mechanics is among the most successful of physical theories. Many of its predictions, especially those of quantum electrodynamics, have been verified with unparalleled precision. So it may seem foolhardy to question its most basic tenet. On the other hand, despite its success, the interpretation of quantum mechanics remains problematic [1]. There is still no generally accepted solution to the problem of reduction of the wave packet. Moreover, there are grave difficulties confronting relativistic quantum theory, difficulties confronting relativistic quantum theory, difficulties which have been circumvented but not eliminated by renormalization theory. There resolution may well require a thorough reappraisal of the basic principles on which the theory is founded. The linearity of quantum mechanics, expressed in the superposition principle is anomalous. Linearity is a common feature of physical theories, but in all other known cases it is an approximation. The range over which linearity holds may be extensive, but is always limited: Maxwell's equations break down for very intense fields and the linearity of space-time itself is a weak-field approximation. Practically all physical phenomena behave nonlinearly when examined over a sufficiently large range of the dynamical parameters that determine their evolution.

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A number of earlier works that have attempted to extend quantum theory in a nonlinear way, there are: The work by Broglie-Bohm nonlinear wave mechanics [2, 3] and the Heisenberg spinor unified field theory [4, 5] until present [6, 7], the developments of various nonlinear theories are always remarkable. Thacker discussed an exact integrability in quantum field theory and statistical systems, which include the nonlinear Schrodinger model and equation [6].

A number of different nonrelativistic models of this kind have been systematically studied by Weinberg, offering also an assessment of the observational limits on such modifications of the Schrödinger equation [8]. Independently, Doebner and Goldin and collaborators have also studied nonlinear modifications of the nonrelativistic Schrödinger equation [9]. This was originally motivated by attempts to incorporate dissipative effects. Later, however, they have shown that classes of nonlinear Schrödinger equations, including many of those considered earlier, can be obtained through nonlinear transformations of the linear quantum mechanical equation. The nonlinear quantum mechanics has a practical importance in different fields, like condensed matter, quantum optics and atomic and molecular physics; even quantum gravity may involve nonlinear quantum mechanics [10–13]. Another important example is in the modern field of quantum computing [14–18].

In this paper, we study the nonlinear Schrödinger equation by adding some higher order terms to the Lagrangian density of Schrödinger field, and obtain some nonlinear Schrödinger equation, in which include the famous Gross-Pitaevskii (GP) equation and the logarithmic form equation.

2 The Lagrangian of Schrödinger Field

The field Lagrangian L is a functional of field amplitude $\psi(\vec{r},t)$. It can usually be expressed as the integral over all space of a Lagrangian density $\mathcal{L}$:

$$L=\int \mathcal{L}(\psi,\nabla\psi,\dot{\psi},t)d^3\vec{r}. \tag{1}$$

The actual field is derived from Hamilton's variational principle:

$$\delta\int_{t_1}^{t_2} Ldt=\delta\int_{t_1}^{t_2}\int \mathcal{L}dtd^3\vec{r}=0, \tag{2}$$

where the infinitesimal variation $\delta\psi$ of ψ is subject to the restrictions

$$\delta\psi(\vec{r},t_1)=\delta\psi(\vec{r},t_2)=0. \tag{3}$$

From (1), (2) and (3), we can obtain the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\psi}-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right)-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\right)=0. \tag{4}$$

If the field Lagrangian density $\mathcal{L}$ is given, we can obtain the classical field equation from (4).

From the point of view of second quantization, the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t)=-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t)+V\psi(\vec{r},t), \tag{5}$$

is a wave equation of classical field $\psi(\vec{r},t)$, and the Schrödinger wave field $\psi(\vec{r},t)$ is a kind of complex field. The equation of complex field $\psi^*(\vec{r},t)$ is

$$-i\hbar\frac{\partial}{\partial t}\psi^*(\vec{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi^*(\vec{r},t) + V\psi^*(\vec{r},t). \tag{6}$$

The Lagrangian density of Schrödinger field may be taken to be

$$\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V\psi^*\psi. \tag{7}$$

Substituting (7) into (4), we can obtain the Schrödinger equations (5) and (6).

Since

$$\frac{\partial\mathcal{L}}{\partial\psi} = -V\psi^*, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^*, \qquad \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)} = -\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}, \tag{8}$$

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = i\hbar\dot{\psi} - V\psi, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_i)} = -\frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x_i}, \tag{9}$$

substituting (8) and (9) into (4), we have

$$-i\hbar\frac{\partial}{\partial t}\psi^* = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*, \tag{10}$$

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{11}$$

The (10) and (11) are Schrödinger equations, and then the Lagrangian density (7) is the Lagrangian density of Schrödinger field.

3 The Nonlinear Schrödinger Equation from the Higher Order Term

The Lagrangian density of Schrödinger filed $\mathcal{L}$ is the function of ψ, $\dot{\psi}$, ψ^*, $\dot{\psi}^*$, $\nabla\psi$ and $\nabla\psi^*$. When it takes the form of (7), which includes the terms $\psi^*\cdot\dot{\psi}$, $\nabla\psi^*\cdot\nabla\psi$ and $\psi^*\cdot\psi$, it can obtain the linear Schrödinger equation (10) or (11).

If the Lagrangian density includes higher order terms, e.g., $\nabla\psi^*\cdot\nabla\psi\cdot\psi^*\cdot\psi$, $(\nabla\psi^*)^2(\nabla\psi)^2$, $\ln(\psi^*\psi)^m$, $\ln\sqrt[n]{\psi^*\psi}$, $(\psi^*\psi)^m$, $\sqrt[n]{\psi^*\psi}$ and so on, we can obtain the nonlinear Schrödinger equation. In the following, we shall add the higher order terms to the Lagrangian density of Schrödinger filed.

(1) adding term $\nabla\psi^*\cdot\nabla\psi\cdot\psi\cdot\psi^*$ to the Lagrangian density $\mathcal{L}$

When we add term $\nabla\psi^*\cdot\nabla\psi\cdot\psi\cdot\psi^*$ to the Lagrangian density, the Lagrangian density of Schrödinger filed becomes

$$\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V\psi^*\psi + a\nabla\psi^*\cdot\nabla\psi\cdot\psi^*\psi, \tag{12}$$

where a is a constant.

Since

$$\frac{\partial\mathcal{L}}{\partial\psi}=-V\psi^*+a\nabla\psi^*\cdot\nabla\psi\cdot\psi^*, \tag{13}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\hbar\psi^*, \tag{14}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}=-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}+a\psi^*\psi\frac{\partial\psi^*}{\partial x_i}, \tag{15}$$

then

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\right)=-\frac{\hbar^2}{2m}\nabla^2\psi^*+a\nabla(\psi^*\psi)\cdot\nabla\psi^*+a\psi^*\psi\cdot\nabla^2\psi^*, \tag{16}$$

substituting (13)–(16) into (4), we have

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-a^*\nabla\psi^*\cdot\nabla\psi\cdot\psi\\+a^*\nabla(\psi^*\cdot\psi)\cdot\nabla\psi+a^*(\psi^*\psi)\nabla^2\psi.\end{aligned} \tag{17}$$

Equation (17) is the nonlinear Schrödinger equation adding higher order term $\nabla\psi^*\cdot\nabla\psi\cdot\psi\cdot\psi^*$ to the Lagrangian density of Schrödinger filed.

(2) adding term $(\nabla\psi^*)^2(\nabla\psi)^2$ to the Lagrangian density $\mathcal{L}$

When we add term $\nabla\psi^*\cdot\nabla\psi\cdot\psi\cdot\psi^*$ to the Lagrangian density, the Lagrangian density of Schrödinger field is

$$\mathcal{L}=i\hbar\psi^*\dot{\psi}-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi-V\psi^*\psi+b(\nabla\psi^*)^2(\nabla\psi)^2, \tag{18}$$

where b is a constant.

Since

$$\frac{\partial\mathcal{L}}{\partial\psi}=-V\psi^*, \tag{19}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\hbar\psi^*, \tag{20}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}=-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}+2b(\nabla\psi^*)^2\left(\frac{\partial\psi}{\partial x_i}\right), \tag{21}$$

then

$$\sum_{i}\frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\right)=-\frac{\hbar^2}{2m}\nabla^2\psi^*+4b\nabla^2\psi^*\nabla\psi^*\cdot\nabla\psi+2b(\nabla\psi^*)^2\nabla^2\psi, \tag{22}$$

substituting (19)–(22) into (4), we have

$$i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi+4b^*\nabla^2\psi\nabla\psi\cdot\nabla\psi^*+2b^*(\nabla\psi)^2\nabla^2\psi^*. \tag{23}$$

Equation (23) is the nonlinear Schrödinger equation adding higher order term $(\nabla\psi^*)^2(\nabla\psi)^2$ to the Lagrangian density of Schrödinger filed.

(3) adding term $\ln(\psi^*\psi)^m$ to the Lagrangian density $\mathcal{L}$

When we add term $\ln(\psi^*\psi)^m$ to the Lagrangian density, the Lagrangian density of Schrödinger field becomes

$$\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V\psi^*\psi + c\ln(\psi^*\psi)^m, \tag{24}$$

where c is a constant.

Since

$$\frac{\partial\mathcal{L}}{\partial\psi} = -V\psi^* + cm\frac{\psi^*}{\psi^*\psi}, \tag{25}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^*, \tag{26}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)} = -\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}, \tag{27}$$

then

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\right) = -\frac{\hbar^2}{2m}\nabla^2\psi^*, \tag{28}$$

substituting (24)–(28) into (4), we have

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi - c^*m\frac{\psi}{\psi^*\psi}. \tag{29}$$

Equation (29) is the nonlinear Schrödinger equation adding higher order term $\ln(\psi^*\psi)^m$ to the Lagrangian density of Schrödinger field.

(4) adding term $\sqrt[n]{\psi^*\psi}$ to the Lagrangian density $\mathcal{L}$

When we add term $\sqrt[n]{\psi^*\psi}$ to the Lagrangian density, the Lagrangian density of Schrödinger field becomes

$$\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V\psi^*\psi + d\sqrt[n]{\psi^*\psi}, \tag{30}$$

where d is a constant.

Since

$$\frac{\partial\mathcal{L}}{\partial\psi} = -V\psi^* + \frac{1}{n}d(\psi^*\psi)^{\frac{1}{n}-1}\cdot\psi^*, \tag{31}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^*, \tag{32}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)} = -\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}, \tag{33}$$

then

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\right) = -\frac{\hbar^2}{2m}\nabla^2\psi^*, \tag{34}$$

substituting (31)–(34) into (4), we have

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-\frac{1}{n}d^*(\psi^*\psi)^{\frac{1}{n}-1}\cdot\psi. \tag{35}$$

Equation (35) is the nonlinear Schrödinger equation adding higher order term $\sqrt[n]{\psi^*\psi}$ to the Lagrangian density of Schrödinger field.

(5) adding term $(\psi^*\psi)^m$ to the Lagrangian density $\mathcal{L}$

When we add term $(\psi^*\psi)^m$ to the Lagrangian density, the Lagrangian density of Schrödinger field becomes

$$\mathcal{L}=i\hbar\psi^*\dot{\psi}-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi-V\psi^*\psi+e(\psi^*\psi)^m. \tag{36}$$

where e is a constant.

Since

$$\frac{\partial\mathcal{L}}{\partial\psi}=-V\psi^*+em(\psi^*\psi)^{m-1}\psi^*, \tag{37}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\hbar\psi^*, \tag{38}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}=-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}, \tag{39}$$

then

$$\sum_i\frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\right)=-\frac{\hbar^2}{2m}\nabla^2\psi^*, \tag{40}$$

substituting (37)–(40) into (4), we have

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-e^*m|\psi|^{2(m-1)}\psi. \tag{41}$$

Equation (41) is the nonlinear Schrödinger equation adding higher order term $(\psi^*\psi)^m$ to the Lagrangian density of Schrödinger field.

(6) adding term $\psi\psi^*\ln(\psi\psi^*)$ to the Lagrangian density $\mathcal{L}$

When we add term $\psi\psi^*\ln(\psi\psi^*)$ to the Lagrangian density, the Lagrangian density of Schrödinger field is

$$\mathcal{L}=i\hbar\psi^*\dot{\psi}-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi-V\psi^*\psi+f\psi\psi^*\ln(\psi\psi^*). \tag{42}$$

Where f is a constant. Since

$$\frac{\partial\mathcal{L}}{\partial\psi}=-V\psi^*+f\psi^*\ln(\psi\psi^*)+f\psi^*, \tag{43}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\hbar\psi^*, \tag{44}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}=-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}, \tag{45}$$

substituting (43)–(45) into (4), we have

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-f^*\ln(|\psi|^2)\psi+f^*\psi. \tag{46}$$

Equation (46) is the nonlinear Schrödinger equation adding higher order term $f\psi\psi^*\ln(\psi\psi^*)$ to the Lagrangian density of Schrödinger field.

(7) adding term $\psi\psi^*\ln\frac{\psi}{\psi^*}$ to the Lagrangian density $\mathcal{L}$

When we add term $\psi\psi^*\ln\frac{\psi}{\psi^*}$ to the Lagrangian density, the Lagrangian density of Schrödinger field becomes

$$\mathcal{L}=i\hbar\psi^*\dot{\psi}-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi-V\psi^*\psi+g\psi\psi^*\ln\frac{\psi}{\psi^*}. \tag{47}$$

Where g is a constant. Since

$$\frac{\partial\mathcal{L}}{\partial\psi}=-V\psi^*+g\psi^*\ln\frac{\psi}{\psi^*}+g\psi^*, \tag{48}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\hbar\psi^*, \tag{49}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}=-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i}, \tag{50}$$

substituting (47)–(50) into (4), we have

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-g^*\left(\ln\frac{\psi}{\psi^*}\right)^*\psi-g^*\psi. \tag{51}$$

Equation (51) is the nonlinear Schrödinger equation adding higher order term $\psi\psi^*\ln\frac{\psi}{\psi^*}$ to the Lagrangian density of Schrödinger field.

When $m=2$, the nonlinear equation (41) is the same as the Gross-Pitaevskii (GP) equation [19, 20]

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-g|\psi|^2\psi. \tag{52}$$

The nonlinear equation (46) is the same as the equation taken in the Logarithmic form in [21]

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-b\psi\ln|\psi|^2. \tag{53}$$

The nonlinear equation (51) is similar as the equation in [19, 20]

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-i\hbar\frac{b}{2m}\left(\ln\frac{\psi}{\psi^*}\right)\psi. \tag{54}$$

In the following, we shall discuss the nonlinear equation (41).

When $m=2$, (41) is

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi-2e^*|\psi|^2\psi=0, \tag{55}$$

and the Lagrangian density of Schrödinger field is

$$\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi - V\psi^*\psi + e(\psi^*\psi)^2. \tag{56}$$

We can determine the constant e by dimension analysis. The Lagrangian density $\mathcal{L}$ of Schrödinger field is the dimension of energy density, i.e., $[E][L]^{-3}$, and ψ dimension is $[L]^{-\frac{3}{2}}$. In (56), the dimension of constant e is $[E][L]^3$, and the dimension of constant $\frac{\hbar^3}{m^2c}$ is $[E][L]^3$. The constant e can be written as

$$e = g\frac{\hbar^3}{m^2c}. \tag{57}$$

Where the constant g is dimensionless. The (55) becomes

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi - g\frac{2\hbar^3}{m^2c}|\psi|^2\psi = 0. \tag{58}$$

Since $\frac{\hbar^3}{m^2c} \ll 1$, the nonlinearly term is very small, i.e., the nonlinearity of Schrödinger (58) is very weak.

When $m = 3$, (41) is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi - 3e^*|\psi|^4\psi = 0, \tag{59}$$

and the Lagrangian density of Schrödinger field is

$$\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi - V\psi^*\psi + e(\psi^*\psi)^3. \tag{60}$$

The dimension of constant e is $[E][L]^6$, and the dimension of constant $\frac{\hbar^6}{m^5c^4}$ is $[E][L]^6$ also. So, the constant e can be written as

$$e = g'\frac{\hbar^6}{m^5c^4}, \tag{61}$$

where the constant g' is dimensionless. The (59) becomes

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi - g'\frac{3\hbar^6}{m^5c^4}|\psi|^4\psi = 0. \tag{62}$$

Obviously, the nonlinearity of Schrödinger equation (62) is weak also.

4 Conclusion

In this paper, we add higher order terms to the Lagrangian density of Schrödinger field and obtain some nonlinear Schrödinger equations. These equations include nonlinear Schrödinger equations proposed by some authors, such as Gross-Pitaevskii (GP) equation and the logarithmic form nonlinear equation. In addition, we prove the coefficient of nonlinear term is very small, i.e., the nonlinearity of Schrödinger equation is weak. We think these nonlinear Schrödinger equations can be used widely in condensed matter and nonlinear fields.

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